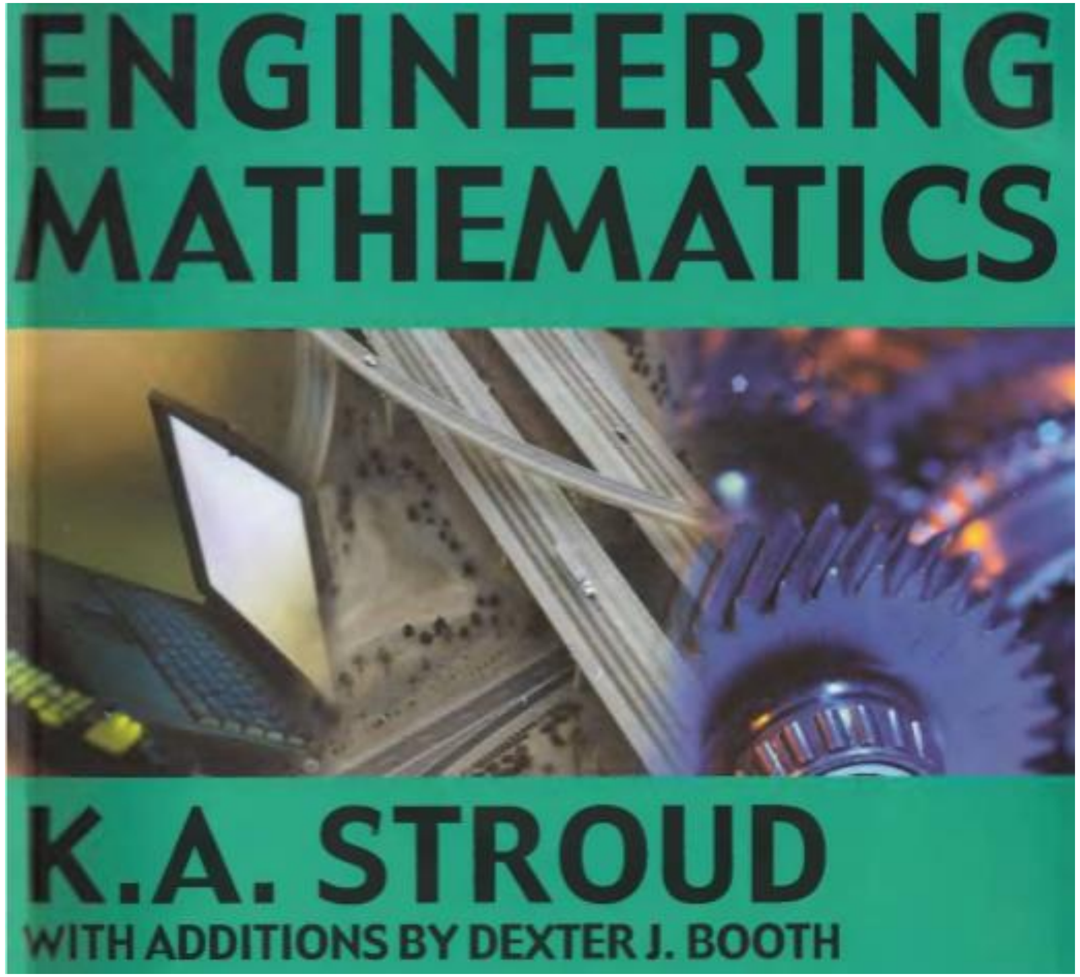

Differential Equation

Introduction

A *differential equation* is a relationship between an independent variable, x, a dependent variable y, and one or more derivatives of y with respect to x.

e.g. $$x^2 \frac{dy}{dx} = y \sin x = 0$$

$$xy \frac{d^2y}{dx^2} + y \frac{dy}{dx} + e^{3x} = 0$$

Differential equations represent dynamic relationships, i.e. quantities that change, and are thus frequently occurring in scientific and engineering problems.

Introduction

The *order* of a differential equation is given by the highest derivative involved in the equation.

$x\dfrac{dy}{dx} - y^2 = 0$ is an equation of the 1st order

$xy\dfrac{d^2y}{dx^2} - y^2 \sin x = 0$ is an equation of the 2nd order

$\dfrac{d^3y}{dx^3} - y\dfrac{dy}{dx} + e^{4x} = 0$ is an equation of the 3rd order

So that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y = \sin 2x$ is an equation of the ……2nd……order.

the highest derivative involved is $\dfrac{d^2y}{dx^2}$.

Formation of Differential Equation

Example 1

Consider $y = A\sin x + B\cos x$, where A and B are two arbitrary constants.
If we differentiate, we get:

$$\frac{dy}{dx} = A\cos x - B\sin x$$

and $$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

which is identical to the original equation, but with the sign changed.

i.e. $\frac{d^2y}{dx^2} = -y \qquad \therefore \quad \frac{d^2y}{dx^2} + y = 0$

This is a differential equation of the ……2nd…… order.

Example 2

Form a differential equation from the function $y = x + \frac{A}{x}$.

We have $y = x + \frac{A}{x} = x + Ax^{-1}$

$$\therefore \quad \frac{dy}{dx} = 1 - Ax^{-2} = 1 - \frac{A}{x^2}$$

From the given equation, $\frac{A}{x} = y - x \quad \therefore \quad A = x(y - x)$

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{x(y-x)}{x^2}$$

$$= 1 - \frac{y-x}{x} = \frac{x - y + x}{x} = \frac{2x - y}{x}$$

$$\therefore \quad x\frac{dy}{dx} = 2x - y$$

This is an equation of the ……1st…… order.

Example 3

Form the differential equation for $y = Ax^2 + Bx$.

We have $$y = Ax^2 + Bx \qquad (1)$$

$$\therefore\ \frac{dy}{dx} = 2Ax + B \qquad (2)$$

$$\therefore\ \frac{d^2y}{dx^2} = 2A \qquad (3) \quad A = \frac{1}{2}\frac{d^2y}{dx^2}$$

Substitute for $2A$ in (2): $\dfrac{dy}{dx} = x\dfrac{d^2y}{dx^2} + B$

$$\therefore\ B = \frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

Substituting for A and B in (1), we have:

$$y = x^2 . \frac{1}{2}\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

$$= \frac{x^2}{2} . \frac{d^2y}{dx^2} + x . \frac{dy}{dx} - x^2 . \frac{d^2y}{dx^2}$$

$$\therefore\ y = x\frac{dy}{dx} - \frac{x^2}{2} . \frac{d^2y}{dx^2}$$

A function with 1 arbitrary constant gives a 1st-order equation. A function with 2 arbitrary constants gives a 2nd-order equation.

and this is an equation of the .....2nd..... order.

Solution of Differential Equation

1. By Direct Integration

If the equation can be arranged in the form $\frac{dy}{dx} = f(x)$, then the equation can be solved by simple integration.

Example 1

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

Then $y = \int (3x^2 - 6x + 5)dx = x^3 - 3x^2 + 5x + C$

i.e. $y = x^3 - 3x^2 + 5x + C$

Value C cannot be determined

As always, of course, the constant of integration must be included. Here it provides the one arbitrary constant which we always get when solving a first-order differential equation.

Example

Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that $y = 3$ when $x = 0$.

First rewrite the equation in the form $\frac{dy}{dx} = \frac{4}{e^x} = 4e^{-x}$.

Then $\quad y = \int 4e^{-x}\,dx = -4e^{-x} + C$

Knowing that when $x = 0$, $y = 3$, we can evaluate C in this case, so that the required particular solution is $y = \ldots\ldots\ldots\ldots$

$$\boxed{y = -4e^{-x} + 7}$$

Solution of Differential Equation

2. By Separating the Variable

If the given equation is of the form $\frac{dy}{dx} = f(x, y)$, the variable y on the right-hand side prevents solving by direct integration. We therefore have to devise some other method of solution.

Let us consider equations of the form $\frac{dy}{dx} = f(x).F(y)$ and of the form $\frac{dy}{dx} = \frac{f(x)}{F(y)}$, i.e. equations in which the right-hand side can be expressed as products or quotients of functions of x or of y.

A few examples will show how we proceed.

Example 1

Solve $\dfrac{dy}{dx} = \dfrac{2x}{y+1}$

We can rewrite this as $(y+1)\dfrac{dy}{dx} = 2x$

Now integrate both sides with respect to x:

$$\int (y+1)\frac{dy}{dx}\,dx = \int 2x\,dx \quad \text{i.e.}$$

$$\int (y+1)\,dy = \int 2x\,dx$$

and this gives $\dfrac{y^2}{2} + y = x^2 + C$

Example 2

Solve $\dfrac{dy}{dx} = (1 + x)(1 + y)$

$$\frac{1}{1+y}\frac{dy}{dx} = 1 + x$$

Integrate both sides with respect to x:

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int (1+x)\,dx \qquad \therefore \quad \int \frac{1}{1+y}\,dy = \int (1+x)\,dx$$

$$\ln(1+y) = x + \frac{x^2}{2} + C$$

The method depends on our being able to express the given equation in the form $F(y).\dfrac{dy}{dx} = f(x)$. If this can be done, the rest is then easy, for we have

$$\int F(y).\frac{dy}{dx}\,dx = \int f(x)dx \qquad \therefore \quad F(y)\,dy = \int f(x)\,dx$$

and we then continue as in the examples.

Solution of Differential Equation

3. Homogeneous Equation – by substituting y=vx

Here is an equation:

$$\frac{dy}{dx} = \frac{x + 3y}{2x}$$

This looks simple enough, but we find that we cannot express the RHS in the form of 'x-factors' and 'y-factors', so we cannot solve by the method of separating the variables.

In this case we make the substitution $y = vx$, where v is a function of x. So $y = vx$. Differentiate with respect to x (using the product rule):

$$\therefore \quad \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Also $$\frac{x + 3y}{2x} = \frac{x + 3vx}{2x} = \frac{1 + 3v}{2}$$

The equation now becomes $v + x\dfrac{dv}{dx} = \dfrac{1 + 3v}{2}$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+3v}{2} - v$$

$$= \frac{1+3v-2v}{2} = \frac{1+v}{2}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+v}{2}$$

The given equation is now expressed in terms of v and x, and in this form we find that we can solve by separating the variables. Here goes:

$$\int \frac{2}{1+v}\,dv = \int \frac{1}{x}\,dx$$

$$\therefore \quad 2\ln(1+v) = \ln x + C = \ln x + \ln A$$

$$(1+v)^2 = Ax$$

But $y = vx \quad \therefore \quad v = \left\{\frac{y}{x}\right\} \quad \therefore \quad \left(1+\frac{y}{x}\right)^2 = Ax$

which gives $(x+y)^2 = Ax^3$

Example 1

Solve $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$

Here, all terms of the RHS are of degree 2, i.e. the equation is homogeneous.
$\therefore$ We substitute $y = vx$ (where v is a function of x)

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\text{and} \quad \frac{x^2 + y^2}{xy} = \frac{x^2 + v^2x^2}{vx^2} = \frac{1 + v^2}{v}$$

The equation now becomes:

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

$$= \frac{1 + v^2 - v^2}{v} = \frac{1}{v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1}{v}$$

Now you can separate the variables and get the result in terms of v and x.

Because

$$\int v \, \mathrm{d}v = \int \frac{1}{x} \mathrm{d}x$$

$$\therefore \quad \frac{v^2}{2} = \ln x + C$$

All that remains now is to express v back in terms of x and y. The substitution we used was $y = vx \quad \therefore \; v = \dfrac{y}{x}$

$$\therefore \quad \frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln x + C$$

$$y^2 = 2x^2(\ln x + C)$$

Solution of Differential Equation

4. Linear equation – use of integrating factor

Consider the equation $\frac{dy}{dx} + 5y = e^{2x}$

This is clearly an equation of the first order, but different from those we have dealt with so far. In fact, none of our previous methods could be used to solve this one, so we have to find a further method of attack.

In this case, we begin by multiplying both sides by e^{5x}. This gives

$$e^{5x}\frac{dy}{dx} + y5e^{5x} = e^{2x}.e^{5x} = e^{7x}$$

We now find that the LHS is, in fact, the derivative of $y.e^{5x}$.

$$\therefore \quad \frac{d}{dx}\left\{y.e^{5x}\right\} = e^{7x}$$

Now, of course, the rest is easy. Integrate both sides with respect to x:

$$\therefore \; y.e^{5x} = \int e^{7x}\,dx = \frac{e^{7x}}{7} + C \quad \therefore \; y = \ldots\ldots\ldots\ldots$$

$$\boxed{y = \frac{e^{2x}}{7} + Ce^{-5x}}$$

Example 1

To solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = x$

If we compare this with $\dfrac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$, we see that in this case

$$P = -1 \text{ and } Q = x.$$

The integrating factor is always $e^{\int P\mathrm{d}x}$ and here $P = -1$.

$\therefore \quad \int P\,\mathrm{d}x = -x$ and the integrating factor is therefore .... $\boxed{e^{-x}}$

Example 2

Solve $x\dfrac{dy}{dx} + y = x^3$

First we divide through by x to reduce the first term to a single $\dfrac{dy}{dx}$

i.e. $\dfrac{dy}{dx} + \dfrac{1}{x}.y = x^2$

Compare with $\left[\dfrac{dy}{dx} + Py = Q\right]$ $\therefore$ $P = \dfrac{1}{x}$ and $Q = x^2$

$IF = e^{\int P dx}$ $\int P\,dx = \int \dfrac{1}{x}\,dx = \ln x$

$\therefore$ $IF = e^{\ln x} = x$ $\therefore$ $IF = x$

The solution is $y.IF = \int Q.IF\,dx$

so $yx = \int x^2.x\,dx = \int x^3\,dx = \dfrac{x^4}{4} + C$ $\therefore$ $xy = \dfrac{x^4}{4} + C$

Example 3

Solve $\dfrac{dy}{dx} + y\cot x = \cos x$

Compare with $\left[\dfrac{dy}{dx} + Py = Q\right] \quad \therefore \begin{cases} P = \cot x \\ Q = \cos x \end{cases}$

$\text{IF} = e^{\int P dx} \qquad \int P\,dx = \int \cot x\,dx = \int \dfrac{\cos x}{\sin x}\,dx = \ln \sin x$

$\therefore \; \text{IF} = e^{\ln \sin x} = \sin x$

$y.\text{IF} = \int Q.\text{IF}\,dx \quad \therefore \; y \sin x = \int \sin x \cos x\,dx = \dfrac{\sin^2 x}{2} + C$

$\therefore \; y = \dfrac{\sin x}{2} + C \operatorname{cosec} x$

Solution of Differential Equation

5. Bernoulli Equation

These are equations of the form:

$$\frac{dy}{dx} + Py = Qy^n$$

where, as before, P and Q are functions of x (or constants).

The trick is the same every time:

(a) Divide both sides by y^n. This gives:

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

(b) Now put $z = y^{1-n}$

so that, differentiating, $\boxed{\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}}$

Example 1

Solve $\dfrac{dy}{dx}+\dfrac{1}{x}y=xy^2$

(a) Divide through by y^2, giving $\boxed{y^{-2}\dfrac{dy}{dx}+\dfrac{1}{x}.y^{-1}=x}$

(b) Now put $z=y^{1-n}$, i.e. in this case $z=y^{1-2}=y^{-1}$

$$z=y^{-1}\quad\therefore\quad\frac{dz}{dx}=-y^{-2}\frac{dy}{dx}$$

(c) Multiply through the equation by (-1), to make the first term $\dfrac{dz}{dx}$.

$$-y^{-2}\frac{dy}{dx}-\frac{1}{x}y^{-1}=-x$$

so that $\dfrac{dz}{dx}-\dfrac{1}{x}z=-x$ which is of the form $\dfrac{dz}{dx}+Pz=Q$ so that you can now solve the equation by the normal integrating factor method. What do you get?

$$\boxed{y=(Cx-x^2)^{-1}}$$

Check the working:

$$\frac{dz}{dx} - \frac{1}{x}z = -x$$

$$\text{IF} = e^{\int P\,dx} \qquad \int P\,dx = \int -\frac{1}{x}\,dx = -\ln x$$

$$\therefore\ \text{IF} = e^{-\ln x} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$z.\text{IF} = \int Q.\text{IF}\,dx \quad \therefore\ z\frac{1}{x} = \int -x.\frac{1}{x}\,dx$$

$$\therefore\ \frac{z}{x} = \int -1\,dx = -x + C$$

$$\therefore\ z = Cx - x^2$$

But $z = y^{-1}$ $\quad \therefore\ \dfrac{1}{y} = Cx - x^2 \quad \therefore\ y = (Cx - x^2)^{-1}$

Revision summary

1 The *order* of a differential equation is given by the highest derivative present.
An equation of *order* n is derived from a function containing n *arbitrary constants*.

2 *Solution of first-order differential equations*

(a) By direct integration: $\dfrac{dy}{dx} = f(x)$
gives $y = \int f(x)\,dx$

(b) By separating the variables: $F(y).\dfrac{dy}{dx} = f(x)$
gives $\int F(y)\,dy = \int f(x)\,dx$

(c) Homogeneous equations: Substituting $y = vx$
gives $v + x\dfrac{dv}{dx} = F(v)$

(d) Linear equations: $\dfrac{dy}{dx} + Py = Q$
Integrating factor, $\text{IF} = e^{\int P dx}$
and remember that $e^{\ln F} = F$
gives $y\text{IF} = \int Q.\text{IF}\,dx$

(e) Bernoulli's equation: $\dfrac{dy}{dx} + Py = Qy^n$
Divide by y^n: then put $z = y^{1-n}$
Reduces to type (d) above.

Exercise

The questions are similar to the equations you have been solving in the Programme. They cover all the methods, but are quite straightforward. Do not hurry: take your time and work carefully and you will find no difficulty with them.

Solve the following differential equations:

1 $x\dfrac{dy}{dx} = x^2 + 2x - 3$

2 $(1+x)^2\dfrac{dy}{dx} = 1 + y^2$

3 $\dfrac{dy}{dx} + 2y = e^{3x}$

4 $x\dfrac{dy}{dx} - y = x^2$

5 $x^2\dfrac{dy}{dx} = x^3 \sin 3x + 4$

6 $x\cos y\dfrac{dy}{dx} - \sin y = 0$

7 $(x^3 + xy^2)\dfrac{dy}{dx} = 2y^3$

8 $(x^2 - 1)\dfrac{dy}{dx} + 2xy = x$

9 $\dfrac{dy}{dx} + y\tanh x = 2\sinh x$

10 $x\dfrac{dy}{dx} - 2y = x^3\cos x$

11 $\dfrac{dy}{dx} + \dfrac{y}{x} = y^3$

12 $x\dfrac{dy}{dx} + 3y = x^2y^2$