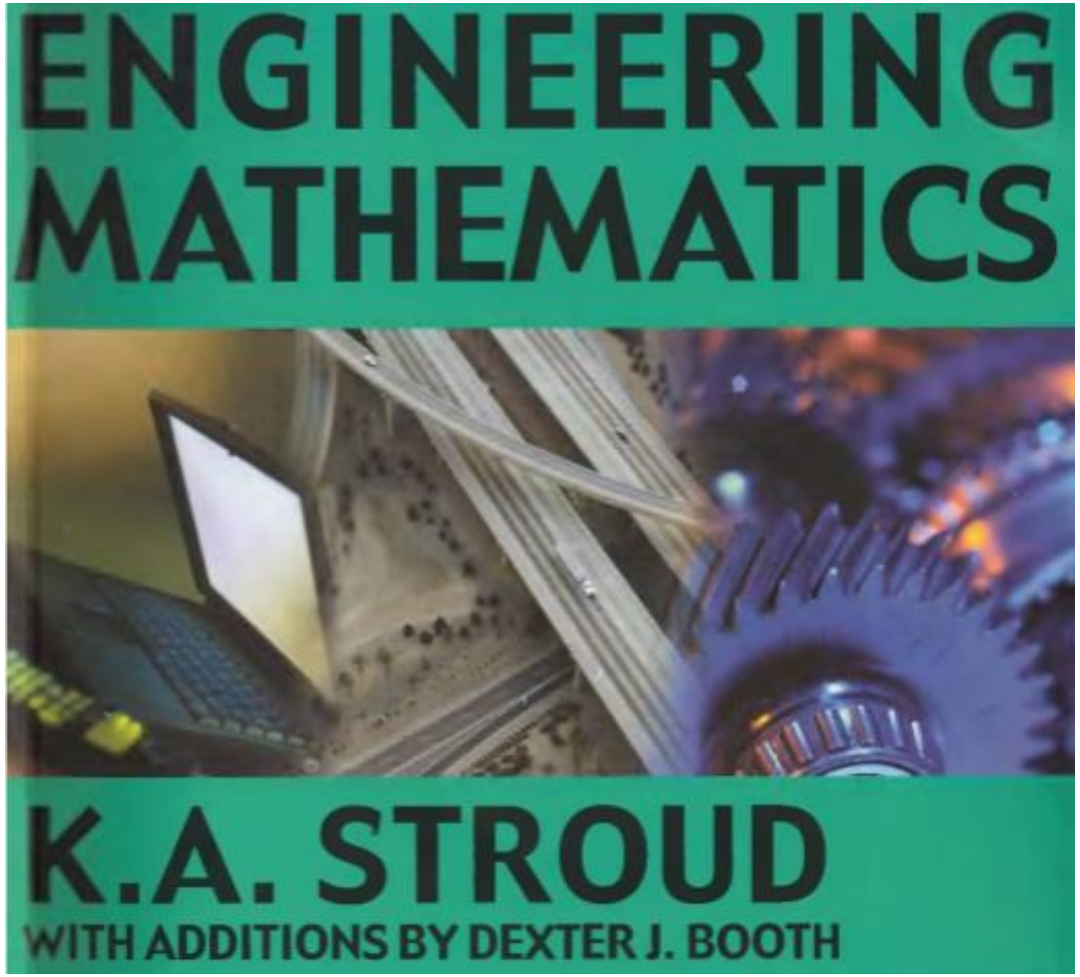


# 2nd Order Differential Equation

Many practical problems in engineering give rise to second-order differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

where $a$, $b$ and $c$ are constant coefficients and $f(x)$ is a given function of $x$. By the end of this Programme you will have no difficulty with equations of this type.

$$\boxed{aAm^2e^{mx}+bAme^{mx}+cAe^{mx}=0}$$

$$y=Ae^{m_1x}+Be^{m_2x}$$

**Example**

For the equation $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$, the auxiliary equation is $2m^2 + 5m + 6 = 0$.

In the same way, for the equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, the auxiliary equation is $\boxed{m^2 + 3m + 2 = 0}$

Since the auxiliary equation is always a quadratic equation, the values of $m$ can be determined in the usual way.

i.e. if $m^2 + 3m + 2 = 0$

$$(m + 1)(m + 2) = 0 \quad \therefore \; m = -1 \text{ and } m = -2$$

$\therefore$ the solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ is

$$y = Ae^{-x} + Be^{-2x}$$

In the same way, if the auxiliary equation were $m^2 + 4m - 5 = 0$, this factorizes into $(m + 5)(m - 1) = 0$ giving $m = 1$ or $-5$, and in this case the solution would be

$$\boxed{y = Ae^{x} + Be^{-5x}}$$

## 1 *Real and different roots* to the auxiliary equation

### Example 1

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Auxiliary equation: $m^2 + 5m + 6 = 0$

$\therefore$ $(m+2)(m+3) = 0$ $\therefore$ $m = -2$ or $m = -3$

$\therefore$ Solution is $y = Ae^{-2x} + Be^{-3x}$

### Example 2

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$$

Auxiliary equation: $m^2 - 7m + 12 = 0$

$(m-3)(m-4) = 0$ $\therefore$ $m = 3$ or $m = 4$

So the solution is $\boxed{y = Ae^{3x} + Be^{4x}}$

## 2 *Real and equal roots* to the auxiliary equation

Let us take $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$

The auxiliary equation is: $m^2 + 6m + 9 = 0$

$\therefore\ (m+3)(m+3) = 0 \quad \therefore\ m = -3$ (twice)

If $m_1 = -3$ and $m_2 = -3$ then these would give the solution $y = Ae^{-3x} + Be^{-3x}$ and their two terms would combine to give $y = Ce^{-3x}$. But every second-order differential equation has two arbitrary constants, so there must be another term containing a second constant. In fact, it can be shown that $y = Kxe^{-3x}$ also satisfies the equation, so that the complete general solution is of the form $y = Ae^{-3x} + Bxe^{-3x}$

i.e. $\quad y = e^{-3x}(A + Bx)$

In general, if the auxiliary equation has real and equal roots, giving $m = m_1$ twice, the solution of the differential equation is

$$y = e^{m_1 x}(A + Bx)$$

**Example 1**

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

Auxiliary equation: $m^2 + 4m + 4 = 0$

$(m + 2)(m + 2) = 0 \quad \therefore \; m = -2$ (twice)

The solution is: $y = e^{-2x}(A + Bx)$

Here is another:

**Example 2**

Solve $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

Auxiliary equation: $m^2 + 10m + 25 = 0$

$(m + 5)^2 = 0 \quad \therefore \; m = -5$ (twice)

$y = e^{-5x}(A + Bx)$

### 3 *Complex roots* to the auxiliary equation

Now let us see what we get when the roots of the auxiliary equation are complex.

Suppose $m = \alpha \pm j\beta$, i.e. $m_1 = \alpha + j\beta$ and $m_2 = \alpha - j\beta$. Then the solution would be of the form:

$$\begin{aligned} y &= Ce^{(\alpha+j\beta)x} + De^{(\alpha-j\beta)x} = Ce^{\alpha x}.e^{j\beta x} + De^{\alpha x}.e^{-j\beta x} \\ &= e^{\alpha x}\{Ce^{j\beta x} + De^{-j\beta x}\} \end{aligned}$$

Now from our previous work on complex numbers, we know that:

$$e^{jx} = \cos x + j\sin x$$

$$e^{-jx} = \cos x - j\sin x$$

$$\text{and that } \begin{cases} e^{j\beta x} = \cos\beta x + j\sin\beta x \\ e^{-j\beta x} = \cos\beta x - j\sin\beta x \end{cases}$$

Our solution above can therefore be written:

$$y = e^{\alpha x}\{C(\cos\beta x + j\sin\beta x) + D(\cos\beta x - j\sin\beta x)\}$$
$$= e^{\alpha x}\{(C+D)\cos\beta x + j(C-D)\sin\beta x\}$$
$$y = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\}$$

where $A = C + D$ and $B = j(C - D)$

$\therefore$ If $m = \alpha \pm j\beta$, the solution can be written in the form:

$$y = e^{\alpha x}\left\{A\cos\beta x + B\sin\beta x\right\}$$

Here is an example: If $m = -2 \pm j3$

then $y = e^{-2x}\left\{A\cos 3x + B\sin 3x\right\}$

Similarly, if $m = 5 \pm j2$ then $y =$ $\boxed{y = e^{5x}[A\cos 2x + B\sin 2x]}$

Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+9y=0$

Auxiliary equation: $m^2+4m+9=0$

$\therefore\ m=\frac{-4\pm\sqrt{16-36}}{2}=\frac{-4\pm\sqrt{-20}}{2}=\frac{-4\pm 2j\sqrt{5}}{2}=-2\pm j\sqrt{5}$

In this case $\alpha=-2$ and $\beta=\sqrt{5}$

Solution is: $y=e^{-2x}(A\cos\sqrt{5}x+B\sin\sqrt{5}x)$